

MODULI SPACE OF CR-PROJECTIVE COMPLEX FOLIATED TORI

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ABSTRACT. We study the moduli space of CR-projective complex foliated tori. We describe it in terms of isotropic subspaces of Grassmannian and we show that it is a normal complex analytic space.

RÉSUMÉ. Nous étudions l'espace des modules des tores feuilletés par des feuilles complexes, plongés de façon CR dans un espace projective. Nous le décrivons en termes des sous-espaces isotropiques de la Grassmannienne et nous prouvons que c'est un espace analytique complexe normale.

1. INTRODUCTION

A *complex foliated torus* or *CR-torus* is defined as $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$, where $\Gamma \subset \mathbb{C}^n \times \mathbb{R}^k$ is a lattice. The natural projection of $\mathbb{C}^n \times \mathbb{R}^k$ on \mathbb{R}^k induces a foliation of $\mathbb{T}^{n,k}$ whose leaves are complex manifolds of dimension n . Hence, $\mathbb{T}^{n,k}$ is a compact Levi-flat manifold.

Complex foliated tori are related to *quasi-Abelian varieties*, i.e. to quotients of \mathbb{C}^n by a discrete subgroup. In fact, any quasi-Abelian variety, as real manifold, is diffeomorphic to the product of a CR-torus and a real linear vector space (see e.g. [1], [2]). For more details and results on quasi-Abelian varieties we refer to [3], [5].

In this paper we consider complex foliated tori endowed with a *polarization* ω , namely ω is a bilinear skew-symmetric form on $\mathbb{R}^{2n} \times \mathbb{R}^k$ taking integral values on $\Gamma \times \Gamma$ and such that its restriction to \mathbb{R}^{2n} is the imaginary part of a positive definite Hermitian form on \mathbb{C}^n . The existence of a polarization on a complex foliated torus $\mathbb{T}^{n,k}$ is equivalent to the CR-embeddability into a projective space, i.e. to the existence of an analytic embedding of $\mathbb{T}^{n,k}$ into a projective space that is holomorphic along the leaves (see [6] and the references included). We will call *CR-projective* such tori. Note that the definition of polarization we take in consideration here is more general of that one considered in [6].

We define the *moduli space* of CR-projective tori and we show that it is a

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normal complex analytic space (see Theorem 3.6). In order to do this, we will present the space of period matrices as a subset of the Grassmannian of n -complex planes in \mathbb{C}^{2n+k} .

The authors would like to dedicate this paper to the memory of Nicolangelo Medori.

2. PRELIMINARIES

Let $\mathbb{C}^n \times \mathbb{R}^k$ be endowed with the natural CR-structure. A *complex foliated torus* or *CR-torus* is a torus

$$\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma,$$

where Γ is a lattice in $\mathbb{R}^{2n+k} = \mathbb{R}^{2n} \times \mathbb{R}^k$, with the natural CR-structure induced by $\mathbb{C}^n \times \mathbb{R}^k$. Note that the projection $\pi : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{T}^{n,k}$ gives rise to a foliation on $\mathbb{T}^{n,k}$, whose leaves are complex manifolds of dimension n . Let (z, t) denote any point of $\mathbb{C}^n \times \mathbb{R}^k$. A *CR-map* $\varphi : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{C}^n \times \mathbb{R}^k$ is a smooth map of the form

$$\varphi(z, t) = (f(z, t), h(t))$$

where f is holomorphic with respect to z . A CR-map between complex foliated tori $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$ and $\mathbb{T}^{n',k} = \mathbb{C}^{n'} \times \mathbb{R}^k / \Gamma'$ is given by a CR-map

$$\tilde{\varphi} : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{C}^{n'} \times \mathbb{R}^k$$

such that $\tilde{\varphi}(\Gamma) \subset \Gamma'$. The following lemma characterizes the CR-maps between complex foliated tori.

Lemma 2.1. *Let $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$ and $\mathbb{T}^{n',k} = \mathbb{C}^{n'} \times \mathbb{R}^k / \Gamma'$ be complex foliated tori and $\varphi : \mathbb{T}^{n,k} \rightarrow \mathbb{T}^{n',k}$ be a CR-map. Then φ is induced by a CR-map $\tilde{\varphi} : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{C}^{n'} \times \mathbb{R}^k$ given by*

$$\tilde{\varphi}(z, t) = (Az + Bt + \beta(t), Ct + \gamma(t)),$$

where $A \in M_{n',n}(\mathbb{C})$, $B \in M_{n',k}(\mathbb{C})$, $C \in M_{k,k}(\mathbb{R})$ and

$$\beta : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{C}^{n'}, \quad \gamma : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{R}^k$$

do not depend on z , are Γ -periodic and $(\beta(0), \gamma(0))$ belongs to Γ' .

Proof. As already remarked, φ is induced by $\tilde{\varphi} : \mathbb{C}^n \times \mathbb{R}^k \rightarrow \mathbb{C}^{n'} \times \mathbb{R}^k$,

$$\tilde{\varphi}(z, t) = (f(z, t), h(t)).$$

such that, $f(z, t)$ is a holomorphic map of z , for any $t \in \mathbb{R}^k$. As for the case of complex tori (see e.g. [4, Chap.1, Proposition 2.1]) we get

$$(1) \quad f(z, t) = A(t)z + B(t),$$

where $A(t)$ and $B(t)$ are a $n' \times n$ complex matrix and a vector of $\mathbb{C}^{n'}$ respectively, both depending on t . Since

$$(2) \quad f(z + \zeta, t + \tau) - f(z, t) \in \Gamma',$$

for any $(z, t) \in \mathbb{C}^n \times \mathbb{R}^k$, $(\zeta, \tau) \in \Gamma \subset \mathbb{C}^n \times \mathbb{R}^k$, we have:

$$(3) \quad \frac{\partial}{\partial t_s}(f(z + \zeta, t + \tau) - f(z, t)) = 0, \quad s = 1, \dots, k.$$

By taking into account (1), then (3) implies that

$$\frac{\partial}{\partial t_s}[(A(t + \tau) - A(t))z + A(t + \tau)\zeta + B(t + \tau) - B(t)] = 0.$$

By a CR-change of coordinates in $\mathbb{C}^n \times \mathbb{R}^k$, we may assume that the lattice Γ contains points of the form $(0, \tau)$.

Hence, evaluating the last expression for $z = 0$ and $\zeta = 0$, we obtain

$$(4) \quad \frac{\partial}{\partial t_s}[B(t + \tau) - B(t)] = 0.$$

Hence

$$(5) \quad \frac{\partial}{\partial t_s}[A(t + \tau)(z + \zeta) - A(t)z] = 0.$$

Then (5) and (4) imply that

$$(6) \quad A^{ij}(t) = \sum_{h=1}^k A_h^{ij} t_h + \alpha^{ij}(t).$$

and

$$(7) \quad B(t) = Bt + \beta(t),$$

where A_h^{ij} , $B = (B^{ih})$ are constant and α^{ij} , β are periodic. Now we are going to show that

$$A_h^{ij} = 0, \quad h = 1, \dots, k, \quad \text{and } \alpha^{ij}, \quad i, j = 1, \dots, n, \text{ are constant.}$$

Differentiating (2) with respect to z and taking onto account (1), we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial z_r}[f(z + \zeta, t + \tau) - f(z, t)] = \\ &= \frac{\partial}{\partial z_r}[(A(t + \tau) - A(t))z + A(t + \tau)\zeta + B(t + \tau) - B(t)]. \end{aligned}$$

Therefore (6) and (7) imply that

$$\frac{\partial}{\partial z_r}[\sum_{h=1}^k (\sum_{j=1}^n A_h^{ij} \tau_h z_j + B^{ih} \tau_h)] = 0.$$

Hence $A_h^{ij} = 0$. By (5) we get

$$\frac{\partial}{\partial t_s}[A(t)\zeta] = 0.$$

Therefore A is constant and the proposition is proved. \square

Let $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$ be a complex foliated torus. Let

$$\gamma_1 = (z_1, t_1), \dots, \gamma_{2n+k} = (z_{2n+k}, t_{2n+k}),$$

$z_1, \dots, z_{2n+k} \in \mathbb{C}^n, t_1, \dots, t_{2n+k} \in \mathbb{R}^k$, be a \mathbb{Z} -basis of Γ . Then

$$\Omega = \begin{pmatrix} z_1 & \cdots & z_{2n+k} \\ t_1 & \cdots & t_{2n+k} \end{pmatrix} \in M_{n+k, 2n+k}(\mathbb{C})$$

is called a *period matrix* of $\mathbb{T}^{n,k}$. Observe that Ω depends on the choice of the \mathbb{Z} -basis of Γ . We have the following

Proposition 2.2. *Let $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$ and $\mathbb{T}'^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma'$ be complex foliated tori. Denote by Ω and Ω' period matrices for $\mathbb{T}^{n,k}$ and $\mathbb{T}'^{n,k}$ respectively. Then $\mathbb{T}^{n,k}$ and $\mathbb{T}'^{n,k}$ are CR-isomorphic if and only if there exists*

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

with $A \in \mathrm{GL}(n, \mathbb{C}), B \in M_{n,k}(\mathbb{C}), C \in \mathrm{GL}(k, \mathbb{R})$, and $P \in \mathrm{GL}(2n+k, \mathbb{Z})$ such that

$$(8) \quad M\Omega = \Omega'P.$$

Proof. If there exist M and P as in (8), then the map $\tilde{\varphi}(z, t) = (Az + Bt, Ct)$ induces a CR-diffeomorphism between $\mathbb{T}^{n,k}$ and $\mathbb{T}'^{n,k}$. Vice versa, if $\varphi : \mathbb{T}^{n,k} \rightarrow \mathbb{T}'^{n,k}$ is a diffeomorphism, then Lemma 2.1 implies that $\tilde{\varphi}(z, t) = (Az + Bt + \beta(t), Ct + \gamma(t))$. Since $(\beta(0), \gamma(0)) \in \Gamma'$, then (8) holds. \square

We will denote by $L_{n,k}$ the group of matrices given by

$$L_{n,k} = \left\{ M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in \mathrm{GL}(n, \mathbb{C}), B \in M_{n,k}(\mathbb{C}), C \in \mathrm{GL}(k, \mathbb{R}) \right\}.$$

Two period matrices Ω and Ω' are said to be *equivalent* if they satisfy condition (8) for some $M \in L_{n,k}, P \in \mathrm{GL}(2n+k, \mathbb{Z})$.

By definition, the *moduli space of complex foliated tori* is the quotient $\mathcal{M}_{n,k}$ of the space of period matrices modulo the equivalence relation (8).

It can be checked that any equivalence class of period matrices has a representative Ω of the following form

$$\Omega = \begin{pmatrix} Z & 0 \\ T & I_k \end{pmatrix}.$$

We will call *adapted* such a period matrix.

3. MODULI OF POLARIZED COMPLEX FOLIATED TORI

Let $\mathbb{T}^{n,k} = \mathbb{C}^n \times \mathbb{R}^k / \Gamma$ be a complex foliated torus.

Definition 3.1. A polarization on $\mathbb{T}^{n,k}$ is given by a skew-symmetric bilinear form ω on $\mathbb{R}^{2n} \times \mathbb{R}^k$ such that:

- i) $\omega|_{\Gamma \times \Gamma}$ takes integral values;
- ii) $\omega|_{\mathbb{R}^{2n} \times \{0\}}$ is the imaginary part of a positive definite Hermitian form on \mathbb{C}^n .

A complex foliated torus $\mathbb{T}^{n,k}$ is said to be *CR-projective* if it can be CR-embedded into a $\mathbb{P}^N(\mathbb{C})$. It turns out that a complex foliated torus $\mathbb{T}^{n,k}$ is CR-projective if and only if there exists a polarization ω on $\mathbb{T}^{n,k}$ (see [6]). For a CR-embedding theorem of a compact Levi-flat manifold of codimension one one can see [9, Theorem 3].

A pair $(\mathbb{T}^{n,k}, \omega)$ is said to be a *polarized complex foliated torus*. Two polarized complex foliated tori $(\mathbb{T}^{n,k}, \omega)$ and $(\mathbb{T}^{n,k}, \omega')$ are said to be *equivalent* if there exists a CR-diffeomorphism $\varphi : \mathbb{T}^{n,k} \rightarrow \mathbb{T}^{n,k}$ such that the restrictions of ω and $\varphi^* \omega'$ to $\mathbb{C}^n \times \{0\}$ coincide.

Remark 3.2. Any polarized complex foliated torus $(\mathbb{T}^{n,k}, \omega')$ is equivalent to $(\mathbb{T}^{n,k}, \omega)$, where ω is represented by

$$\begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}$$

and $\mathbb{T}^{n,k}$ has an adapted period matrix. To show this, set $\eta = \omega'|_{\mathbb{C}^n \times \{0\}}$. Let

$$\Omega' = \begin{pmatrix} Z' & W' \\ T' & R' \end{pmatrix}$$

be a period matrix for $\mathbb{T}^{n,k}$. By changing the order of the columns of Ω' we may assume that R' is invertible. Then, by acting on the left with

$$M = \begin{pmatrix} A & -AW'R'^{-1} \\ 0 & R'^{-1} \end{pmatrix},$$

where $A \in \mathrm{U}(n)$, we obtain that Ω is an adapted period matrix and that the restriction of $\varphi^* \omega'$ to $\mathbb{C}^n \times \{0\}$ is still η , since $A \in \mathrm{U}(n)$ (where φ is the CR-diffeomorphism represented by the matrix M).

As for the complex case, there exist $d_1, \dots, d_n \in \mathbb{N}$ with $d_i | d_{i+1}$, $i = 1, \dots, n-1$, such that the polarization ω can be represented by the matrix

$$(9) \quad \begin{pmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where Δ is the diagonal matrix having diagonal entries d_1, \dots, d_n (see e.g. [7, Chap. VI, Proposition 1.1], [8, Lemma p.204]).

Let

$$\begin{aligned} F_{n,k} &= \left\{ M = \begin{pmatrix} A & 0 \\ 0 & I_k \end{pmatrix} \mid A \in \mathrm{U}(n) \right\}, \\ H_{n,k} &= \left\{ P = \begin{pmatrix} \alpha & 0 \\ \beta & I_k \end{pmatrix} \mid \alpha \in \mathrm{Sp}(2n, \mathbb{Z}), \beta \in \mathrm{M}_{k,2n}(\mathbb{Z}) \right\}. \end{aligned}$$

Let $\mathbb{T}^{n,k}$ and $\mathbb{T}'^{n,k}$ be two complex foliated tori endowed with the same polarization ω . As observed in Remark 3.2, we may assume that ω has the canonical form given by (9) and Ω and Ω' are adapted period matrices of $\mathbb{T}^{n,k}$ and $\mathbb{T}'^{n,k}$ respectively.

We say that Ω and Ω' are *equivalent* if there exist $M \in F_{n,k}$ and $P \in H_{n,k}$ such that

$$(10) \quad M\Omega = \Omega'P.$$

Definition 3.3. *The moduli space of CR-projective complex foliated tori is the quotient $\mathcal{M}_{n,k}^\omega$ of the space of period matrices modulo the equivalence relation (10).*

We are going to describe explicitly the space of period matrices. Let Ω be an adapted period matrix for the polarized complex foliated torus $(\mathbb{T}^{n,k}, \omega)$. Denote by $\gamma_1, \dots, \gamma_{2n+k}$ the column vectors of Ω . Set

$$\begin{cases} J\gamma_i &= \gamma_{n+i}, & i = 1, \dots, n, \\ J\gamma_{n+i} &= -\gamma_i, & i = 1, \dots, n. \end{cases}$$

Then J defines a complex structure on the real vector space V spanned by $\gamma_1, \dots, \gamma_{2n}$. These vectors give an \mathbb{R} -isomorphism f between V and \mathbb{R}^{2n} by setting $f(\gamma_i) = e_i, i = 1, \dots, 2n$. In this way (V, J) is isomorphic to (\mathbb{R}^{2n}, J_0) . Then

$$g(u, v) = f^*\omega(u, Jv)$$

is a J -invariant inner product on V . Then, by setting

$$L = V^{1,0} = \{x - iJx \mid x \in V\},$$

we get that the complex n -plane $L \subset \mathbb{C}^{2n+k}$ satisfies

$$L \cap \overline{L} = \{0\}, \quad f^*(\hat{\omega}|_{\mathbb{C}^{2n}}) = 0,$$

where $\hat{\omega}$ is the complexification of ω . Hence, we have proved the following

Proposition 3.4. *The space of period matrices of CR-projective complex foliated tori is given by*

$$\mathcal{U}_{n,k}^\omega = \{L \in \mathrm{Gr}_{\mathbb{C}}(n, 2n+k) \mid L \cap \overline{L} = \{0\}, L \text{ is } f^*\hat{\omega}\text{-isotropic}\}.$$

As a consequence of the previous proposition we get the following

Corollary 3.5. *The space of period matrices of CR-projective complex foliated tori is a Kähler manifold of complex dimension $\frac{1}{2}n(n+1+2k)$.*

Now we can describe the moduli space (for the case of Abelian varieties see e.g. [4], [7]). We have the following

Theorem 3.6. *The moduli space $\mathcal{M}_{n,k}^\omega$ of CR-projective complex foliated tori is given by*

$$\mathcal{M}_{n,k}^\omega \simeq \mathcal{U}_{n,k}^\omega / H_{n,k}.$$

Any discrete subgroup of $H_{n,k}$ acts properly discontinuously on $\mathcal{U}_{n,k}^\omega$. In particular, $\mathcal{M}_{n,k}^\omega$ is a normal analytic space.

Proof. By Proposition 3.4, we immediately get that $\mathcal{M}_{n,k}^\omega \simeq \mathcal{U}_{n,k}^\omega / H_{n,k}$. Now, we show that any discrete subgroup G of $H_{n,k}$ acts properly discontinuously on $\mathcal{U}_{n,k}^\omega$. Consider two compact sets K_1 and K_2 in $\mathcal{U}_{n,k}^\omega$. Let

$$M = \begin{pmatrix} \alpha & 0 \\ \beta & I_k \end{pmatrix} \in G$$

such that $MK_1 \cap K_2 \neq \emptyset$. Let $V_M \in MK_1 \cap K_2$ and set $V'_M = M^{-1}V_M$. Denote by J and J' be the corresponding complex structures on V_M and V'_M , respectively. Then we have

$$\omega(Ju, Jv) = \omega(u, v), \forall u, v \in V_M, \quad \omega(u, Ju) > 0, \forall u \neq 0$$

and the same holds for V'_M . Take a symplectic basis $\mathcal{B} = \{\gamma_1, \dots, \gamma_{2n}\}$ of V_M and let $\mathcal{B}' = \{\gamma'_1 = M^{-1}\gamma_1, \dots, \gamma'_{2n} = M^{-1}\gamma_{2n}\}$. Then, with respect to the bases \mathcal{B} and \mathcal{B}' of V_M and V'_M respectively (denoting with the same letters the matrices for J , J' and ω), we have $J' = \alpha J \alpha^{-1}$ and, consequently, ${}^t J \omega J = \omega$, ${}^t J' \omega J' = \omega$. Therefore, ${}^t J \omega$ and ${}^t J' \omega$ are symmetric and positive definite matrices. Hence, there exist two orthogonal matrices Q, Q' such that

$${}^t J \omega = {}^t Q D Q, \quad {}^t J' \omega = {}^t Q' D' Q',$$

where D and D' are diagonal and positive definite matrices. By taking into account the previous relations, we get $D' = {}^t S D S$, where $S = Q \alpha Q'$, so that S varies in a compact set. Therefore, α lies in a compact set. By the definition of the action, it can be easily checked that also β varies in a compact set. Hence G is finite. \square

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